

# **G. Birkhoff's problem in irreversible quantum dynamics**

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## **Abstract**

We find a complete invariance upto cocycle conjugacy for an extremal element in the convex set of unital trace preserving completely positive map on matrix algebra over the field of complex numbers. As an application we prove that any extremal element in the set of unital trace preserving completely positive map is also extremal in the convex set of unital completely positive maps.

## 1 Introduction:

We start with a brief history of this problem. An  $n \times n$  matrix  $S = (s_j^i : 1 \leq i, j \leq n)$  is called doubly stochastic if

- (a) all entries  $s_j^i \geq 0$ ;
- (b) all row sums and column sums are equal to 1 i.e.  $\sum_j s_j^i = 1$  for all  $i$  and  $\sum_i s_j^i = 1$  for all  $j$ .

It is obvious that the set  $\mathcal{S}_n$  of all doubly stochastic matrices forms a compact convex subset in  $R^{n^2}$ . A permutation  $\pi$  is an one to one map of the indices  $\{1, 2, \dots, n\}$  onto themselves. The associated permutation matrix  $S_\pi$  is defined by  $s_j^i = 1$  if  $j = \pi(i)$  otherwise 0. Clearly  $S_\pi \in \mathcal{S}_n$ . It is simple also to check that  $S_\pi$  is an extremal element in  $\mathcal{S}_n$ . Conversely D. König [Ko] and G. Birkhoff [Bi1] proves that an extremal point of  $\mathcal{S}_n$  is a permutation matrix  $S_\pi$  for some permutation  $\pi$  on the set  $\{1, 2, \dots, n\}$ . By Carathéodory theorem it follows that all doubly stochastic matrices are convex combination of permutation matrices. However this representation is not unique in general i.e.  $\mathcal{S}_n$  is not a simplex. G. Birkhoff in his book [Bi2] asked for an extension of this problem in infinite dimension.

D. G. Kendall [Ke] settled down this conjecture affirmatively as follows. Let the index set to be a countable infinite set and  $\mathcal{S}$  be the convex set of doubly stochastic matrices and  $\mathcal{P}_\pi$  be the collection of permutation matrices i.e. matrices having exactly one unit element in each row and column, all other entries being equal to zero.  $\mathcal{S}$  is viewed as a subset of infinite dimensional matrices  $\mathcal{S}$  with entries whose rows and columns have uniform bounded  $l^1$  norms:

$$\sup_i \sum_j |s_j^i| < \infty, \quad \sup_j \sum_i |s_j^i| < \infty$$

$\mathcal{S}$  is equipped with the coarsest topology such that the linear maps  $l_j^i(s) = s_j^i$ ,  $l^i(s) = \sum_j s_j^i$  and  $l_j(s) = \sum_i s_j^i$  are continuous and thus  $\mathcal{S}$  so equipped become a topological space. In such a topology  $\mathcal{P}$  is not a compact subset and however D. G.

Kendall and J. C. Kiefer proved that  $\mathcal{S}$  is equal to closer of the convex combination of permutation matrices where closer is taken in the coarsest topology described above.

Within the framework of quantum mechanics of irreversible processes, one major problem is to investigate the same problem extending the scope to doubly stochastic maps on a non-commutative algebra of observable namely a  $C^*$  algebra or a von-Neumann algebra. As a first step we investigate this problem when  $\mathcal{A}$  is a matrix algebra over the field of complex numbers. We refer readers to a recent conference note by Musat and Haagerup [MuH] for results on extremal points in the compact convex set

$$CP_\phi = \{\tau : \mathcal{A} \rightarrow \mathcal{A}, \text{ CP map}, \tau(1) = 1, \phi \circ \tau = \phi\}$$

where  $\mathcal{A}$  is a finite dimensional matrix algebra over the field of complex numbers i.e.  $\mathcal{A} = M_n(C)$  and  $\phi$  is the normalized trace on  $\mathcal{A}$ .

It is quite some time now that it is known that there are extreme points in  $CP_\phi$  other than automorphisms in case dimension of matrices  $n$  is more than equal to three [LS,KM]. However a complete description or characterization along the line of G. Birkhoff remains missing and most approach towards this problem takes clue from classical work of M. D. Choi [Ch] on characterization of extremal elements of unital completely positive maps. As a first step we investigate this problem when  $\mathcal{A}$  is a matrix algebra over the field of complex numbers and characterize conjugacy class of it's extreme points.

Here we adopt a different approach first largely following a dynamical system's point of view and make use of it's general results [Mo2] in order to describe topologically the interior and boundary points of the compact convex set  $CP_\phi = \{\tau : \mathcal{A} \rightarrow \mathcal{A}, \text{ CP map}, \tau(1) = 1, \phi\tau = \phi\}$ , where  $\phi$  is a faithful state on  $\mathcal{A} = M_n(C)$ . An extremal element always lies in a convex face made of completely irreducible projections determining a resolution of identity. Thus the theory gives a recursive method to identify extremal elements in  $CP_\phi$  in lower dimensional faces. This recursive method gave rises also the problem when two extremal elements are

cocycle conjugate?

To that end we investigate in details operator spaces associated with a unital CP map and obtain our main result in section 3 by proving a complete order isomorphism property between operator spaces of two such CP maps provided their basic data matrices are unitary or anti-unitary conjugate.

In section 4 we extend Arveson-Hann-Banach extension theorem to lift state preserving property from operator space to its minimal  $C^*$ -algebra and thus obtain a complete order isomorphism property on the minimal  $C^*$ -algebra containing those operators spaces. We have achieved this adopting a dynamical systems point of view of a given CP map. Thus we arrive at our main result by Jordan-Wigner's theorem. As an application we prove that any extremal element in the set of unital ( trace preserving ) CP maps is also an extremal element in the set of unital CP maps on matrix algebra.

Associated Kolmogorov's type of dilation theory [KuM] plays no role in fixing these extremal elements here. Thus methods used here are striking different from recent approach [MuH]. We note also that very little modification is needed to include the case when  $\phi$  is not a trace but just a faithful state on the algebra of matrices over complex field. In such a case one can use a criteria similar to Landau-Streater type using Tomita's Modular relation. We also here note that a claim in the paper [MW] is not in harmony with the second line of our abstract. Further the method that we have adopted here seems to put some light when  $\mathcal{A}$  is a type- $II_1$  factor.

I thank M. B. Ruskai for an invitation to participate in a workshop on " Quantum Information Theory " at Institute Mittag-Leffler organized during fall 2010 and also make me aware about developments in [MuH] and [MW].

## 2 Extremal completely positive unital maps:

Let  $\mathcal{A}$  be a von-Neumann algebra acting on a real or complex separable Hilbert space. A linear map  $\tau : \mathcal{A} \rightarrow \mathcal{A}$  is called positive if  $\tau(x) \geq 0$  for all  $x \geq 0$ . Such a map is automatically bounded with norm  $\|\tau\| = \|\tau(I)\|$ . Such a map  $\tau$  is called completely positive [St] ( $CP$ ) if  $\tau \otimes I_n : \mathcal{A} \otimes M_n \rightarrow \mathcal{A} \otimes M_n$  is positive for each  $n \geq 1$  where  $\tau \otimes I_n$  is defined by  $(x_j^i) \rightarrow (\tau(x_j^i))$  with matrix entries  $(x_j^i)$  are elements in  $\mathcal{A}$ . In this paper we will only consider unital completely positive maps i.e.  $\tau(I) = I$ . Further we assume  $\tau$  to be normal. We will use notation  $CP$  for the convex set of unital completely positive normal maps on  $\mathcal{A}$ .

Let  $\phi$  be a normal state on  $\mathcal{A}$ . An element  $\tau \in CP$  is called  $\phi$  invariant if  $\phi\tau = \phi$ . We denote  $CP_\phi$  for set elements in  $CP$  those are  $\phi$  invariant. Further an  $\phi$ -invariant element  $\tau \in CP$  is called *ergodic* if there exists no non-trivial  $\tau$ -invariant projection. An ergodic element  $\tau$  is called aperiodic if  $\tau$  admits no non-trivial periodic projection i.e. if  $\{E_k : 0 \leq k \leq n\}$  is a family of mutually orthogonal projections in  $\mathcal{A}$  so that  $\sum_{0 \leq k \leq n-1} E_k = I$  and  $\tau(E_k) = E_{k+1}$  for  $0 \leq k \leq n-1$  where  $E_n = E_0$  then  $n = 1$ . Theorem 2.6 in [Mo2] says that aperiodic elements are strongly mixing if  $\mathcal{A}$  is a type-I von-Neumann algebra over a complex field with center completely atomic. Though the same result still holds if  $\mathcal{A}$  were over real field, we will not use it here. Ergodic elements in  $CP_\phi$  are denoted in the following by  $CP_\phi^e$ .

An unital CP map on  $\mathcal{A}$  admits a representation  $\tau(x) = \sum_k v_k x v_k^*$  where  $\{v_k : k \geq 1\}$  be a family of contractions such that  $\sum_k v_k v_k^* = 1$ . However for a given element  $\tau$ , such a choice for a family of operators is not unique. However if  $\mathcal{A}$  is algebra of all bounded operators in a real or complex separable Hilbert space, the vector space  $\mathcal{L}_\tau$  generated by  $\{v_k : k \geq 1\}$  is determined uniquely by  $\tau$ . Further we can give an inner product  $\langle\langle \cdot, \cdot \rangle\rangle$  on  $\mathcal{L}_\tau$  and choose  $v_k$  such that  $\langle\langle v_k, v_j \rangle\rangle = 0$  for  $j \neq k$ . Thus in case  $I \in \mathcal{L}_\tau$  we can write  $\tau = \lambda I_{\mathcal{A}} + (1 - \lambda)\tau_0$  for some  $\lambda > 0$  and  $\tau_0$  is an element in  $CP_\phi$  with  $I \notin \mathcal{L}_{\tau_0}$ , where  $I_{\mathcal{A}}(x) = x$  is the identity map on  $\mathcal{A}$ .

Further if  $\mathcal{A}$  is a matrix algebra, we can choose a finite family of linearly independent elements as  $\mathcal{L}_\tau$  is finite dimensional. For further details we include a ready reference [Ar] and also [ChE],[EvL], [BR] as a general reference on completely positive maps on a  $C^*$  and von-Neumann algebras.

The base with respect to the identity map  $I_{\mathcal{A}}$  is defined to be the set of elements  $\tau$  in the boundary of  $CP_\phi$  that lies on the rare end in the line passing through  $I_{\mathcal{A}}$  and  $\tau$ .

We define

$$CP_\phi = \{\tau \in CP : \tau(1) = 1, \phi\tau = \phi\}$$

and

$$CP_\phi^e = \{\tau \in CP_\phi : \tau \text{ is ergodic} \}.$$

$CP_\phi$  is a convex set. An element  $\tau$  is called extremal if  $\tau$  is extremal in  $CP_\phi$  i.e. if  $\tau = \lambda\tau_1 + (1 - \lambda)\tau_0$  for some  $\tau_0, \tau_1 \in CP_\phi$  and  $0 < \lambda < 1$  then  $\tau_0 = \tau_1$ .

In particular the classical situation discussed above corresponds to the commutative algebra  $l_\infty$  over a finite or countable index set and  $\phi$  is the counting measure. Note that we assumed  $\phi$  to be state and thus the theory won't include countable infinite situation within the frame work discussed here. This we defer for future investigation. More generally a finite dimensional algebra  $\mathcal{A}$  is isomorphic to direct sums of finitely many matrix algebras. Thus to begin with now we consider  $\mathcal{A}$  to be a matrix algebra of  $n \times n$  matrices over  $C$  and aim to describe it's extreme elements  $CP_\phi^e$ .

Let  $\mathcal{A}$  to be  $n \times n$  matrix algebra over the field  $\mathbb{R}$  or  $\mathbb{C}$ .  $CP_\phi$  is a compact subset of an euclidean space. By Carathéodory theorem an element in  $CP_\phi$  is a convex combination of finitely many extremal elements. The identity map  $I_{\mathcal{A}}(x) = x$  is an extremal element in  $CP_\phi$  ( if  $I_{\mathcal{A}} = \lambda I_1 + (1 - \lambda)I_2$  then  $(1 - e)I_k(e)(1 - e) = 0$  for any projection  $e \in \mathcal{A}$  and thus  $I_k = I_{\mathcal{A}}$  for  $k = 1, 2$  ).  $I$  is non-ergodic unless  $\mathcal{A} = \mathbb{C}$ . Other trivial non-ergodic extremal elements are  $\phi$ -invariant automorphisms.

At this point we note that  $\tau$  is extremal if and only if  $\alpha\tau\beta$  is extremal, where  $\alpha, \beta$  are  $\phi$ -invariant automorphisms.

A subset  $F$  of  $CP_\phi$  is called *face* if  $\tau = \lambda\tau_1 + (1-\lambda)\tau_0$  for some  $\tau \in F$ ,  $\tau_0, \tau_1 \in CP_\phi$  and  $0 < \lambda < 1$  then  $\tau_0, \tau_1 \in F$ . Thus each extremal element is a face in its own right. Further both  $F_1 \cap F_2$  and  $F_1 \cup F_2$  are faces if  $F_1$  and  $F_2$  are faces in  $CP_\phi$ . Note that a face need not be a convex set. Further for a face  $F$ , it is elementary to check that convex hull of  $F$  need not be a face.

A simple application of Hahn-Banach separating hyper-plane theorem shows that an element in the boundary of  $CP_\phi$  is contained in a convex face of the boundary. One important property if a face  $F$  is also convex, then extreme points of  $F$  are also extreme points of  $CP_\phi$ . Thus in order to describe extremal elements of  $CP_\phi$ , we will be interested in finding out enough lower dimensional convex faces.

**PROPOSITION 2.1:** Let  $\phi$  be a faithful normal state on a von-Neumann algebra  $\mathcal{A}$  and  $CP_\phi$  be the set of normal completely positive unital  $\phi$ -invariant maps on  $\mathcal{A}$ . Then the following statements are true:

- (a) Both  $CP_\phi^e$  and  $CP_\phi$  are non-empty convex set; The closure of  $CP_\phi^e$  is equal to  $CP_\phi$ ;
- (b) The complement of  $CP_\phi^e$  is a face in  $CP_\phi$ ;
- (c) Further if  $\mathcal{A}$  is a finite type-I von Neumann algebra ( factor corresponds  $\mathcal{A}$  to be a matrix algebra ) then the following hold:
  - (i)  $CP_\phi^e$  is an open convex contractible subset of  $CP_\phi$ ;
  - (ii) An interior element in  $CP_\phi$  is aperiodic;
  - (iii) The set of aperiodic elements in the boundary of  $CP_\phi$  is non-empty;
  - (iv) Let  $\mathcal{A}$  be a finite type-I factor i.e.  $\mathcal{A}$  is an algebra of  $n \times n$  matrices over  $\mathbb{R}$  or  $\mathbb{C}$  and  $\tau$  be ergodic. Then  $\tau$  is either strongly mixing ( equivalently Kolmogorov's property ) or there exists an automorphism  $\alpha$  on  $\mathcal{A}$  such that  $\tau\alpha$  is a non-ergodic element.

**PROOF:** That  $CP_\phi^e$  is non-empty follows as CP map  $x \rightarrow \phi(x)$  is an element in  $CP_\phi^e$ .

For the last part of (a) we fix any two elements  $\tau \in CP_\phi^e$  and  $\eta$  in the complement of  $CP_\phi^e$ ; We claim that  $\lambda\tau + (1 - \lambda)\eta \in CP_\phi^e$  for all  $0 < \lambda \leq 1$ . Fix such an  $\lambda$  and let  $E$  be an invariant projection for the convex combination. Then  $(1 - E)[\lambda\tau(E) + (1 - \lambda)\eta(E)](1 - E) = (1 - E)E(1 - E) = 0$  and so by positive property of each maps we get  $(1 - E)\tau(E)(1 - E) = 0$  as  $\lambda > 0$ . Hence  $\tau(E) \leq E$ . As  $1 - E$  is also an invariant element for the convex combination, we also get  $\tau(1 - E) \leq (1 - E)$ . Thus we get  $\tau(E) = E$  and by ergodic property of  $\tau$  we get  $E$  is either 0 or 1. Thus for each  $1 \geq \lambda > 0$ , the map is ergodic. By taking limit  $\lambda \rightarrow 0$ , we verify our claim.

That  $CP_\phi^e$  is a convex set in  $CP_\phi$  follows along the same line described above as for  $\tau, \eta \in CP_\phi^e$  and  $0 \leq \lambda \leq 1$ , if we have  $\lambda\tau(E) + (1 - \lambda)\eta(E) = E$  then either  $E\tau(1 - E)E = 0$  or  $E\eta(1 - E)E = 0$ . Thus either  $\tau(E) = E$  or  $\eta(E) = E$ . Hence  $E$  is either 0 or 1.

This completes the proof of (b) modulo the face property of the complement of  $CP_\phi^e$ . To that end let  $\tau$  be a non-ergodic element and  $\tau = \lambda\tau_1 + (1 - \lambda)\tau_0$  for some  $\lambda \in (0, 1)$  and  $\tau_0, \tau_1 \in CP_\phi$ . Thus there exists a non-zero projection  $E \in \mathcal{A}$  so that  $\tau(E) = E$ . Thus  $0 = (1 - E)E(1 - E) = \lambda(1 - E)\tau_1(E)(1 - E) + (1 - \lambda)(1 - E)\tau_0(E)(1 - E)$ . Since each element in the sum is non negative and  $\lambda \in (0, 1)$ , we get  $(1 - E)\tau_1(E)(1 - E) = 0 = (1 - E)\tau_0(E)(1 - E)$ . Thus  $\tau_0(E) \leq E$  and  $\tau_0(E) \leq E$ . Interchanging the role of  $E$  and  $1 - E$ , we get  $\tau_1(E) \geq E$  and  $\tau_0(E) \geq E$ . This shows that both  $\tau_0$  and  $\tau_1$  are non-ergodic.

For the first statement (i) of (c) we will show that complement of  $CP_\phi^e$  is a closed set. Let  $\tau_n : n \geq 1$  be a convergent sequence of elements in the complement and  $\tau$  be the limit in the weak\* topology ( which is same as norm topology since  $\mathcal{A}$  is finite dimensional ). Then there exists a sequence of non-trivial projections  $E_n$  so that  $\tau_n(E_n) = E_n$  for  $n \geq 1$ .  $\mathcal{A}$  being finite type-I factor, compactness of the closed



unit ball in  $\mathcal{A}$  is used to extract a sub-sequence say  $E_{n_k}$  so that  $E_{n_k} \rightarrow E$  in the weak\* topology. Since each projection  $E_n$  is non-trivial, we get  $\text{tr}(E_n) \geq 1$  and so  $E$  is also non-trivial.  $\mathcal{A}$  being finite, weak\* topology coincide with norm topology on the unit ball. Thus  $\tau_n(E_n) \rightarrow \tau(E)$  as  $n \rightarrow \infty$  in the weak\* topology. Hence we get  $\tau(E) = E$ . Since  $E$  is non-trivial  $\tau$  is also an element in the complement of  $CP_\phi^e$ . The denseness follows as  $\tau_\lambda = \lambda\tau + (1-\lambda)\phi \in CP_\phi^e$  for  $\lambda \in (1, 0]$  for any  $\tau \in CP_\phi$  and  $\tau_\lambda(x) \rightarrow \tau(x)$  in weak\* topology for any  $x \in \mathcal{A}$  as  $\lambda \rightarrow 1$ .

Let  $\tau$  be an interior point. Since  $I$  is an extremal element in  $CP_\phi$ ,  $\tau \neq I$ . We will prove that  $\tau \in CP_\phi^e$ . Suppose not. Then the line passing through  $\tau$  and  $I$  will intersect an unique point at the rare end in the boundary of  $CP_\phi$  i.e. there exists an unique element  $\beta$  in the boundary such that  $\tau = \lambda I + (1-\lambda)\beta$  for some  $0 < \lambda < 1$ .  $\tau$  now being a non-ergodic element,  $\beta$  is also a non-ergodic element. Thus each element in the line joining  $\beta$  and  $I$  is a non-ergodic element. But the set of ergodic elements being convex, is a connected contractible open set dense in  $CP_\phi$ . This brings a contradiction as  $CP_\phi$  is homotopic to a closed unit ball in  $R^m$  and line joining  $\tau$  and  $I$  is homotopic to a line joining two distinct points in the boundary of the closed unit ball in  $R^m$ .  $CP_\phi^e$  being dense in  $CP_\phi$ , we can get a polygonal line in  $CP_\phi^e$  around the line segment of non-ergodic elements. This brings a contradiction as  $CP_\phi^e$  is contractible. Thus  $\tau = \lambda I + (1-\lambda)\beta$  for some ergodic element  $\beta$  in the boundary of  $CP_\phi$  and  $\lambda \in (0, 1)$ . Now we will prove aperiodic property for  $\tau$ . Let  $\{E_n : n \geq 1\}$  be such a family of projections for  $\tau$ . Then  $E_{n+1} = \tau(E_n) = \lambda E_n + (1-\lambda)\beta(E_n)$  and so  $0 = E_n E_{n+1} E_n = \lambda E_n + (1-\lambda)E_n \beta(E_n) E_n \geq \lambda E_n \geq 0$  and thus  $\lambda = 0$  which contradicts as  $\lambda \in (0, 1)$ . This completes the proof of the statement (ii) of (c).

So far we have proved that non-ergodic elements are in the boundary and some elements in the boundary could be as well ergodic, in fact the set of aperiodic elements in the boundary is non-empty.  $\phi$  is a CP map and is not extremal element. Further  $I \in \mathcal{L}_\phi$  and thus the line segment joining  $I$  and  $\phi$  will meet an element in the boundary of  $CP_\phi$  at the rare end. We write  $\phi = \lambda I + (1-\lambda)\tau$  for some element  $\tau$  in

the boundary of  $CP_\phi$  and  $\lambda \in (0, 1)$ . We claim that  $\tau$  is aperiodic. Ergodic property of  $\tau$  follows trivially. For aperiodic property, let  $(E_k : 0 \leq k \leq n)$  be a family of orthogonal projections such that  $\tau(E_k) = E_{k+1}$  where we set notation  $E_{n+1} = E_0$ . Then  $\phi(E_k)I = \lambda E_k + (1 - \lambda)E_{k+1}$ . Since  $\lambda \in (0, 1)$ , we conclude that  $n = 1$  and thus  $E_0 = I$ . This completes the proof of (iii).

To prove (iv) we recall, if  $\tau$  is not strongly mixing but ergodic, then there exists a non-trivial partition of  $I$  into orthogonal projections  $(E_k : 0 \leq k \leq d)$  with  $d \geq 1$  such that  $\tau(E_k) = E_{k+1}$  where  $E_{d+1} = E_0$ . Since  $\tau$  is trace preserving we have  $\text{rank}(E_k) = \text{trace}(E_k) = \text{trace}(E_{k+1}) = \text{rank}(E_{k+1})$ . Thus we can construct an automorphism  $\alpha$  such that  $\alpha(E_k) = E_{k+1}$ . Now consider the element  $\tau\alpha^{-1} \in CP_\phi$  which preserves each projection  $E_k$  and thus non ergodic. ■

We sum up our main results obtained so far in the following proposition.

**PROPOSITION 2.2:** Let  $\mathcal{A}$  be a finite type-I von-Neumann algebra and  $\phi$  be a faithful state on  $\mathcal{A}$ . Then the following statements are true:

- (a) Non ergodic elements lie on the boundary of  $CP_\phi$  and form a face in  $CP_\phi$ ;
- (b) A proper periodic ergodic element is cocycle conjugate to a non-ergodic element in  $CP_\phi$ .

**PROOF:** Since any interior element is ergodic we have (a) modulo face property. To that end let  $\tau$  be a non-ergodic element and  $\tau = \lambda\tau_1 + (1 - \lambda)\tau_0$  for some  $\lambda \in (0, 1)$  and  $\tau_0, \tau_1 \in CP_\phi$ . Thus there exists a non-zero projection  $E \in \mathcal{A}$  so that  $\tau(E) = E$ . Thus  $0 = (1 - E)E(1 - E) = \lambda(1 - E)\tau_1(E)(1 - E) + (1 - \lambda)(1 - E)\tau_0(E)(1 - E)$ . Since each element in the sum is non negative and  $\lambda \in (0, 1)$ , we get  $(1 - E)\tau_1(E)(1 - E) = 0 = (1 - E)\tau_0(E)(1 - E)$ . Thus  $\tau_0(E) \leq E$  and  $\tau_1(E) \leq E$ . Interchanging the role of  $E$  and  $1 - E$ , we get  $\tau_1(E) \geq E$  and  $\tau_0(E) \geq E$ . This shows that both  $\tau_0$  and  $\tau_1$  are non-ergodic.

Proof of (b) is given in Proposition 2.1 (iv). ■

Last proposition says very little about extremal elements  $\tau \in CP_\phi$  those are

strongly mixing. In case  $\mathcal{A} = M_n(\mathcal{I})$  and  $n \geq 3$ , extremal elements are known to exist [LS] which is strongly mixing [see example 2 in this text]. Thus topological consideration as described above said very little about extreme points those are strongly mixing. Our main goal now is to characterize such extremal elements. To that end we fix two unital CP maps  $\tau, \eta \in CP_\phi$  with equal numerical index and fix  $\tau(x) = \sum_{1 \leq k \leq d} v_k x v_k^*$  and  $\eta(x) = \sum_{1 \leq k \leq d} l_k x l_k^*$  where  $(v_k)$  and  $(l_k)$  are family of linearly operators such that  $\sum_k v_k v_k^* = 1$  and  $\sum_k l_k l_k^* = 1$ .

**PROPOSITION 2.3:** The following holds:

- (a) The matrix  $((v_i v_j^*))$  as element in  $\mathcal{A} \otimes M_d(\mathcal{I})$  is a projection if and only if  $\tau$  is a doubly stochastic matrix i.e.  $\tau \in CP_\phi$ , where  $\phi$  is the normalized trace.
- (b) If  $\tau, \eta$  are elements in  $CP_\phi$  then there exists unitary operators  $u \in \mathcal{A}$  and  $(w_k^j)$  in  $\mathcal{A} \otimes M_d(\mathcal{I})$  such that

$$\sum_k u^* v_k^* w_j^k = l_j^*$$

Further the group  $\mathcal{G} = \{(u, W) : u, W \text{ are unitaries}\}$  acting on the set  $\mathcal{V} = \{v = (v_1, v_2, \dots, v_d) : \sum_i v_i v_i^* = \sum_i v_i^* v_i = I\}$  transitively where the group  $(u, W) \circ (u', W') = (uu', W'W)$ .

**PROOF:** If part follows by direct computation as the doubly stochastic property says  $v^* v = I_n$  where we have written  $v^* = (v_1^*, v_2^*, \dots, v_d^*)$  as row vector and  $((v_i v_j^*)) = v v^*$ . Conversely projection property says that  $v_k (\sum_i v_i^* v_i) v_j^* = v_k v_j^*$  for all  $k, j$  and thus we get  $P^3 = P^2$  where we take  $P = \sum_i v_i v_i^*$  which is a positive operator with eigen values 0 or 1. Thus  $P$  is a projection. As  $tr(P) = tr(\sum v_i v_i^*) = n$ , rank of  $P$  is equal to  $n$ , we conclude that  $P = I_n$ .

For (b), we check that  $rank(vv^*) = tr(vv^*) = n$  and thus for two doubly stochastic elements  $\tau, \eta$  we have two projections  $((v_i v_j^*))$  and  $((l_i l_j^*))$  in  $\mathcal{A} \otimes M_d(\mathcal{I})$  with same rank. Thus there exists a unitary element  $W \in \mathcal{A} \otimes M_d(\mathcal{I})$  so that  $W^* v v^* W = l l^*$ . We write  $W = ((w_j^i))$  with  $w_j^i \in \mathcal{A} = M_n(\mathcal{I})$ ,  $1 \leq i, j \leq d$ . and set  $u : l_j^* f \rightarrow \sum_k v_k^* w_j^k f$  for any  $f \in \mathcal{I}^n$  and  $1 \leq j \leq d$ . Since the map is inner product

preserving, we get a well defined unitary map on  $\mathcal{K}^n$ . That  $(u, W)$  acts on the set  $\mathcal{V}$  needs a simple computation using trace property. This completes the proof. ■

The group  $\mathcal{G}$  though acting on the set  $\mathcal{V}$ , it does not have a natural action on  $CP_\phi$ . However the subgroup  $\mathcal{G}_0$  consists of elements  $(w \otimes \lambda)$ ,  $(\lambda_j^i)$  and  $w$  are unitary matrices in  $M_d(C)$  and  $M_n(C)$  has a natural action on  $CP_\phi$  and an extremal element under the map goes to another extremal element. The deep mathematical question that we ask now how far the converse is true? As a first step we start with a well known example of an extremal point where we note that  $\tau$  and  $\tilde{\tau}$  are not in the same orbit of the group  $\mathcal{G}_0$ .

**EXAMPLE 2.4:** [LS] We consider the following elements  $v_1, v_2 \in M_4(\mathcal{K})$ .

$$v_1 = \begin{pmatrix} 1 & , & 0, & 0, & 0 \\ 0 & , & 0, & 0, & 0 \\ 0 & , & 0, & \frac{1}{\sqrt{2}}, & 0 \\ 0 & , & 0, & 0, & \frac{1}{\sqrt{2}} \end{pmatrix},$$

$$v_2 = \begin{pmatrix} 0 & , & 0, & 0, & 0 \\ 0 & , & 1, & 0, & 0 \\ 0 & , & 0, & \frac{1}{\sqrt{2}}, & 0 \\ 0 & , & 0, & 0, & \frac{i}{\sqrt{2}} \end{pmatrix},$$

So we have  $v_1 v_1^* = D(1, 0, \frac{1}{2}, \frac{1}{2})$ ,  $v_1 v_2^* = D(0, 1, \frac{1}{2}, -\frac{i}{2})$ ,  $v_2 v_1^* = D(0, 1, \frac{1}{2}, \frac{i}{2})$  and  $v_2 v_2^* = D(0, 1, \frac{1}{2}, \frac{1}{2})$ . For any given matrix  $\lambda = (\lambda_j^i) \in M_2(C)$ ,  $\sum_{i,j} \lambda_j^i v_i v_j^* \geq 0$  if and only if  $\lambda_i^i \geq 0$ ,  $i = 1, 2$ ,  $\lambda_1^1 + \lambda_2^2 + 2\operatorname{Re}(\lambda_2^1) \geq 0$  and  $\lambda_1^1 + \lambda_2^2 + 2\operatorname{Re}(i\lambda_2^1) \geq 0$ . It is simple to show that  $\sum \lambda_j^i v_i v_j^* \geq 0$  whenever  $\lambda = (\lambda_j^i) \geq 0$ . However the above relation says that converse is false. Thus the injective unital map from  $M_2(C) \rightarrow D_4$ ,  $(4 \times 4)$  diagonal matrices given by  $\lambda \rightarrow \sum \lambda_j^i v_i v_j^*$  is not an order isomorphism onto map between two operator spaces ( here they are  $C^*$ -algebras ). Though the map is contractive, inverse map is not so.

The element  $g = (W, u) \in \mathcal{G}$  with  $W = ((w_j^i))$  and  $w_j^i = \delta_j^i w_i$  with  $w_1 = I$  and  $w_2 = \text{diagonal}(1, 1, 1, -1)$  takes  $v = (v_1, v_2)$  to  $v^* = (v_1^*, v_2^*)$ . We check that conjugation action  $\mathcal{J}(z_1, z_2, z_3, z_4) = (\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4)$  takes  $\mathcal{J}v_1\mathcal{J} = v_1^*$  and  $\mathcal{J}v_2\mathcal{J} = v_2^*$ . The operator spaces generated by the two sets of vectors  $\{v_i v_j^* : 1 \leq i, j \leq 2\}$  and  $\{v_i^* v_j : 1 \leq i, j \leq 2\}$  are order isomorphic and they are conjugated by anti-unitary operator  $\mathcal{J}$ .

However we will show in the following that there exists no elements in  $\mathcal{G}_0$  which takes  $v$  to  $v^*$ . We will show it by contradiction. Let  $u, w$  be two unitary operator such that  $uv_i w^* = \sum_j \lambda_j^i v_j^*$  where  $(\lambda_j^i)$  be an unitary matrix. Thus  $u\mathcal{L}_v^2 u^* = \mathcal{L}_{v^*}^2$ . Without loss of generality we assume  $u$  commutes with each projections  $|e_i\rangle\langle e_i|$ ,  $1 \leq i \leq 4$  otherwise we modify  $u$  by pre-multiplying a unitary matrix which permutes the basis. Thus in particular we have  $uv_1 u^* = v_1$  and  $uv_2 u^* = v_2$  for some unitary matrix  $u$ . Since  $\{v_i v_j^* : 1 \leq i, j \leq 2\}$  spans all diagonal matrices, we conclude that  $u$  is also diagonal. Similarly we can modify  $w$  if needed to ensure that  $wv_i w^* = v_i$  to ensure that  $w$  is also diagonal. Hence  $\{v_1, v_2\}$  are elements in the span of  $v_1^*, v_2^*$ . Since  $v_1 = v_1^*$ , we get  $v_2$  is an element in the span of  $v_1^*$ . This contradicting our starting assumption as  $v_2$  and  $v_2^*$  are linearly independent. This example forces us to conclude that  $\mathcal{G}_0$  is not enough and needs more affine maps on  $CP_\phi$ . To that end we have a simple observation. ■

**PROPOSITION 2.5:** Let  $\mathcal{J}_1, \mathcal{J}_2$  be two anti-unitary operators on  $C^n$ . Then the map  $\tau \rightarrow \tau^{\mathcal{J}_1, \mathcal{J}_1}$  is an affine one to one map on  $CP_\phi$  where  $\tau^{\mathcal{J}_1, \mathcal{J}_2}(x) = \mathcal{J}_2 \tau(\mathcal{J}_1 x \mathcal{J}_1) \mathcal{J}_2$ .

**PROOF:** For an anti-unitary operator  $\mathcal{J}$  we have  $\mathcal{J}^* = \mathcal{J}$  and  $\mathcal{J}^2 = I$  where by definition  $\mathcal{J}^*$  is the conjugate linear map defined by  $\langle \mathcal{J}^* f, g \rangle = \overline{\langle f, \mathcal{J}g \rangle}$ , where inner product is taken conjugate linear in the second variable. Thus  $(\mathcal{J}_1 v_k \mathcal{J}_2)^* = \mathcal{J}_2 v_k^* \mathcal{J}_1$ . Thus  $\tau^{\mathcal{J}_1, \mathcal{J}_2}$  is an unital CP map. That the element is also  $\phi$  preserving follows trivially by the trace property and  $\mathcal{J}_1^2 = 1 = \mathcal{J}_2^2$ . ■

We enlarge the groups  $\mathcal{G}$  as the collection of elements

$$\mathcal{G}' = \{(u, W) : u, W \text{ are together unitary or anti-unitary}\}$$

and  $\mathcal{G}_0$  as

$$\mathcal{G}'_0 = \{(u, W) \in \mathcal{G}' : W = (w_j^i) : w_j^i = \lambda_j^i w\}$$

where  $u, ((\lambda_j^i))$  are together either unitary matrices or anti-unitary matrices.

It is evident that the group  $\mathcal{G}'_0$  acts naturally on  $CP_\phi$  and each group element in  $\mathcal{G}'_0$  acts on  $CP_\phi$  as an affine one to one map. We aim now to address whether  $\mathcal{G}'_0$  acts transitively on the set of extremal elements in  $CP_\phi$  with numerical index  $d$ .

To that end we first recall here quickly without proof M D Choi-Landau-Streater's criteria for an element  $\tau \in CP_\phi$  to be extremal.

**THEOREM 2.6:** Let  $\tau(x) = \sum_{1 \leq k \leq d} v_i x v_i^*$  be an unital CP map on  $\mathcal{A}$ . Then

(a)  $\tau$  is extremal in  $CP$  if and only if the elements  $((v_i v_j^*))$  is linearly independent i.e.

$$\sum_{1 \leq i, j \leq d} \lambda_j^i v_i v_j^* = 0$$

for some scalars  $\lambda_j^i \in C$  if and only if  $\lambda_j^i = 0$  for all  $1 \leq i, j \leq d$ .

(b) If  $\tau$  is also an element in  $CP_\phi$  then  $\tau$  is extremal in  $CP_\phi$  if and only if the elements  $((v_i v_j^*))$  and  $((v_i^* v_j))$  are bi-independent (linear) i.e.

$$\sum_{1 \leq i, j \leq d} \mu_j^i v_i v_j^* = 0, \quad \sum_{1 \leq i, j \leq d} \mu_j^i v_j^* v_i = 0$$

for some scalars  $\mu_j^i$  if and only if  $\mu_j^i = 0$  for all  $1 \leq i, j \leq d$ . In other words the elements  $\{v_i v_j^* \oplus v_j^* v_i : 1 \leq i, j \leq d\}$  in the vector space  $\mathcal{A} \oplus \mathcal{A}$  are linearly independent;

**PROOF :** For  $\lambda = (\lambda_j^i)$ ,  $\sum \lambda_j^i v_i v_j^* \oplus v_j^* v_i = 0$  if and only if  $\sum \lambda_j^i v_i v_j^* = 0$  and  $\sum \lambda_j^i v_j^* v_i = 0$  as  $f \oplus g = f \oplus 0 + 0 \oplus g$  for all  $f, g \in C^n$ . ■

For an unital CP map  $\tau(x) = \sum_{1 \leq k \leq d} v_k x v_k^*$  with  $\{v_k : 1 \leq k \leq d\}$  linearly independent, we set notation  $\mathcal{L}_\tau$  for the linear span of  $\{v_k : 1 \leq k \leq d\}$ . The subspace is independent of the choice that we make to represent  $\tau$ . We also set self-adjoint operator space  $\mathcal{L}_\tau^2$  for the linear span of  $\{v_i v_j^* : 1 \leq i, j \leq d\}$  and also note that  $\mathcal{L}_\tau^2$  is independent of the choice that we make to represent  $\tau$  ( if each  $(w_i)$  is an element in the linear span of  $(v_i)$  then  $\{w_i w_j^* : 1 \leq i, j \leq d\}$  are in the linear span of  $\{v_i v_j^* : 1 \leq i, j \leq d\}$  ). A simple consequence of Choi's criteria says that for an extremal element  $\tau$  in the set of unital CP maps, there exists a unique element  $\eta$  in the set of unital CP maps so that  $\mathcal{L}_\tau = \mathcal{L}_\eta$ . For two such extremal elements in unital CP map with same index, an isomorphism between their operator spaces induces an isomorphism on the matrix algebra of  $(d \times d)$  matrices. Two such elements are cocycle conjugate by an element in  $\mathcal{G}'_0$  if and only if there exists an order-isomorphism implemented by a unitary or anti-unitary operator so that the induced isomorphism on matrix algebra is also an order isomorphic map. A valid question that we can ask: whether such an order isomorphism holds in general when associated operator spaces are order isomorphic? Further valid question whether two such extremal elements give rise to order isomorphic operator spaces?

Further for an element  $\tau \in CP_\phi$ , we set operator space  $\mathcal{L}_{\tau, \bar{\tau}}^2$  spanned by elements  $\{v_i v_j^* \oplus v_j^* v_i : 1 \leq i, j \leq d\}$  in  $\mathcal{A} \oplus \mathcal{A}$ . It is also obvious that the operator space is independent of representation that we use for  $\tau$ .

**PROPOSITION 2.7:** Let  $\tau$  be an extremal element in  $CP_\phi$ . Then there exists a unique element  $\eta \in CP_\phi$  so that  $\mathcal{L}_\eta = \mathcal{L}_\tau$ .

**PROOF:** Let  $\tau(x) = \sum_k v_k x v_k^*$  be a minimal representation. Let  $\eta$  be another element in  $CP_\phi$  so that  $\mathcal{L}_\eta = \mathcal{L}_\tau$  and we write  $\eta(x) = \sum_{1 \leq k \leq d} l_k x l_k^*$  and  $\mathcal{L}_\eta = \mathcal{L}_\tau$ . We choose  $\lambda = (\lambda_j^i)$  so that  $l_k = \sum_j \lambda_j^k v_j$  as  $\mathcal{L}_\eta = \mathcal{L}_\tau$ . Since  $\sum_k v_k^* v_k = \sum v_k v_k^* = 1$  and also  $\sum_k l_k^* l_k = \sum_k l_k l_k^* = 1$  we get

$$\sum_{j,j'} (\sum_k \bar{\lambda}_j^k \lambda_{j'}^k - \delta_{j'}^j) v_j^* v_{j'} = 0$$

$$\sum_{j,j'} (\sum_k \bar{\lambda}_j^k \lambda_{j'}^k - \delta_{j'}^j) v_{j'} v_j^* = 0$$

Since  $\tau$  is an extremal element we get  $\lambda \in GL_d(C)$  by Theorem 2.6 (b) and thus  $\eta = \tau$ . ■

Two such extremal elements in  $CP_\phi$  are  $\mathcal{G}'_0$  conjugate if and only if their associated operator spaces  $\mathcal{A}_{\tau, \bar{\tau}}^2$  are order-isomorphically conjugated either by a unitary or anti-unitary operator such that the induced map on the matrix coefficients are also order isomorphic. Once more a valid question that one can ask whether any two extremal elements of same index give rise to order isomorphic operator spaces? If so then whether there exists order isomorphism so that induced isomorphism on the matrix coefficients are also order isomorphic?

We are aiming to investigate the question when two extremal elements in  $CP_\phi$  with equal numerical index are cocycle conjugate. Proposition 2.7 says that we need to verify whether their respective linear operator subspaces are co cycle conjugate. In particular our interest lies whether such a statement is true when they are extremal elements in  $CP_\phi$ . As an motivation we discuss now few examples in the following which in particular says that two strongly mixing elements with equal numerical index need not be cocycle conjugate.

**EXAMPLE 2.8:** Let  $\mathcal{A} = M_2(C)$  i.e.  $2 \times 2$ -matrix with entries in complex numbers and  $\phi$  be the normalize trace. As non trivial projections are one dimensional, we have only one kind of faces namely  $\mathcal{F}_{(1,1)}$ . Without loss of generality we assume that  $|e_1 \rangle \langle e_1|$  and  $|e_2 \rangle \langle e_2|$  are  $\tau \in \mathcal{F}_{(1,1)}$ -invariant and if we write  $\tau(x) = \sum_k v_k x v_k^*$  with elements  $\{v_k : k \geq 1\}$  linearly independent, we get  $\{v_k, v_k^* : k \geq 1\}'' \subseteq \{|e_1 \rangle \langle e_1|, |e_2 \rangle \langle e_2|\}' = \{|e_1 \rangle \langle e_2|, |e_2 \rangle \langle e_1|\}''$ . Thus  $k$  is at most two. We write  $v_1 = \lambda_1 |e_1 \rangle \langle e_1| + \mu_1 |e_2 \rangle \langle e_2|$  and  $v_2 = \lambda_2 |e_1 \rangle \langle e_1| + \mu_2 |e_2 \rangle \langle e_2|$  then  $\lambda = (\lambda_1, \lambda_2), \mu = (\mu_1, \mu_2)$  are unit vec-



tors in  $\mathcal{L}^2$ . Note that  $\tau(|e_1\rangle\langle e_2|) = \langle\langle \lambda, \mu \rangle\rangle |e_1\rangle\langle e_2|$ . Thus there exists an one to one and onto correspondence between non-ergodic unital CP maps with the closed unit disc in the complex plane. Any point at the boundary corresponds to  $\lambda$  that is a scalar multiple of  $\mu$  and thus  $v_1 = \theta v_2$ . So  $k = 1$  and  $\tau(x) = vxv^*$  and  $v$  is unitary. On the other hand any point in the interior say  $z \in \mathcal{L}$  with  $|z| < 1$  the completely positive map  $\tau_z$  can be written as convex combination of two extremal points (though not unique). Thus  $\tau_z$  is not an extremal element if  $z = \langle\langle \lambda, \mu \rangle\rangle$  and  $|z| < 1$ . This classifies all the non-ergodic completely positive maps on  $\mathcal{A} = M_2(\mathcal{L})$  with the normalized tracial state  $\phi$  up to conjugacy class. This was known for quite some time and for further details on related results we refer to [MuH]. ■

**EXAMPLE 2.9:** In this example we consider  $\mathcal{A} = M_3(\mathcal{L})$  and aim to describe extreme points in the faces  $\mathcal{F}_{(1,1,1)}$ . The set of elements in  $\mathcal{F}_{(1,1,1)}$  is completely parametrized by complex numbers  $z_1, z_2, z_3 \in \{z : |z| \leq 1\}$  so that the matrix

$$f_{z_1, z_2, z_3} = \begin{pmatrix} 1 & z_1 & z_3 \\ \bar{z}_1 & 1 & z_2 \\ \bar{z}_3 & \bar{z}_2 & 1 \end{pmatrix}$$

is non-negative definite. We can use Schur's method to check that  $f_{z_1, z_2, z_3}$  is non-negative definite if

$$\begin{pmatrix} 1 & z_2 \\ \bar{z}_2 & 1 \end{pmatrix} - \begin{pmatrix} |z_1|^2 & \bar{z}_1 z_3 \\ z_1 \bar{z}_3 & |z_3|^2 \end{pmatrix}$$

is non-negative. i.e.

$$(1 - |z_1|^2)(1 - |z_3|^2) - |z_2 - z_1 \bar{z}_3|^2 \geq 0$$

i.e.

$$1 - |z_1|^2 - |z_2|^2 - |z_3|^2 - 2\operatorname{Re}(\bar{z}_2 \bar{z}_3 z_1) \geq 0$$

Thus we conclude  $\mathcal{F}$  is parametrized by the following compact convex subset of  $\mathcal{L}^3$

$$\{(z_1, z_2, z_3) : |z_1|, |z_2|, |z_3| \leq 1, f(z_1, z_2, z_3) = 1 - |z_1|^2 - |z_2|^2 - |z_3|^2 - 2\operatorname{Re}(\bar{z}_2 \bar{z}_3 z_1) \geq 0\}.$$

and the zero set  $\{(z_1, z_2, z_3) : f(z_1, z_2, z_3) = 0\}$  is the boundary. Here all extreme points in  $\mathcal{F}_{(1,1,1)}$  are automorphisms like in the previous case. ■

**EXAMPLE 2.10:** In this example we consider  $\mathcal{A} = M_3(\mathbb{R})$  and aim to describe extreme points in the faces  $\mathcal{F}_{(1,1,1)}$ . The set of elements in  $\mathcal{F}_{(1,1,1)}$  is parametrized by the following compact convex subset of  $\mathbb{R}^3$   $\{(x_1, x_2, x_3) : |x_1|, |x_2|, |x_3| \leq 1, f(x_1, x_2, x_3) = 1 - |x_1|^2 - |x_2|^2 - |x_3|^3 - 2x_1x_2x_3 \geq 0\}$ . This set is a swollen solid tetrahedron except six edges those are kept fixed. Vertices are  $(1, 1, -1), (1, -1, 1), (-1, 1, 1)$  and  $(-1, -1, -1)$  and they correspond to automorphisms. Thus all the points on the boundary except the points in the interior of six edges are extreme points. Only those four vertices in a face  $f_{(e_1, e_2, e_3)}$  correspond to automorphisms on  $\mathcal{A}$ . Thus there are infinitely many extremal elements in  $CP_\phi$  those are not automorphisms. Besides two such such extremal elements with numerical index 3 need not be cocycle conjugate. Thus the problem over real fields seems far more complicated then over complex field. ■

**EXAMPLE 2.11 [LS]:** We consider the following standard ( irreducible ) representation of Lie algebra  $SO(3)$  in  $C^3$

$$l_x = 2^{-\frac{1}{2}} \begin{pmatrix} 0 & , & 1, & 0 \\ 1 & , & 0, & 1 \\ 0 & , & 1, & 0 \end{pmatrix},$$

$$l_y = 2^{-\frac{1}{2}} \begin{pmatrix} 0 & , & -i, & 0 \\ i & , & 0, & -i \\ 0 & , & i, & 0 \end{pmatrix},$$

$$l_z = \begin{pmatrix} 1 & , & 0, & 0 \\ 0 & , & 0, & 0 \\ 0 & , & 0, & -1 \end{pmatrix}.$$

where  $[l_x, l_y] = il_z$  and set  $\tau(x) = \sum_i v_i x v_i^*$ , where  $v_1 = 2^{-\frac{1}{2}}l_x, v_2 = 2^{-\frac{1}{2}}l_y, v_3 = 2^{-\frac{1}{2}}l_z$ . The element  $\tau$  is also an extremal element in the convex set of unital  $CP$  maps. However complex conjugation does not leads to another extremal elements in  $CP_\phi$  as in Example 2.4 as  $\mathcal{J}v_1\mathcal{J} = v_1, \mathcal{J}v_2\mathcal{J} = -v_2$  and  $\mathcal{J}v_3\mathcal{J} = v_3$ . ■

**EXAMPLE 2.12:** In this example we consider once more  $\mathcal{A}_0 = M_3(\mathcal{C})$ . The numerical index of an unital CP map take values 1, 2, 3, 4, 5, 6, 7, 8, 9. We fix a complex number  $\theta$  such that  $\theta^3 = 1$  and set unitary operators  $u, v$  defined by

$$ue_k = \lambda^k e_k, \quad ve_k = e_{\text{mod}_3[k+1]}$$

where  $\{e_k : 0 \leq k \leq 2\}$  is an orthonormal basis for  $\mathcal{C}^3$ . It is simple to check that  $uv = \lambda vu$  and trace of  $u$  and  $v$  are 0. Further we set  $u_{i,j} = u^i v^j$  for all  $0 \leq i, j \leq 2$  and by definition  $u_{0,0} = I$  and  $u_{i,0} = u^i$  and  $u_{0,j} = v^j$ . It is a simple computation to check that trace of  $(u_{i,j}^* u_{i',j'}) = \lambda^{j'(i'-i)} v^{j'-j} u^{i'-i}$  is zero if  $(i, j) \neq (i', j')$ . Thus  $u_{i,j}$  is a basis for the vector space of  $n \times n$  complex matrices. We enumerate this family of unitary matrices as  $\{v_k : 0 \leq k \leq 8\}$  such that  $v_0 = 1, v_1 = v$  and  $v_2 = u$ .

We fix an integer  $2 \leq m \leq 8$  and consider the map  $\tau(x) = \frac{1}{m} \sum_{1 \leq k \leq m} v_k x v_k^*$ . Since the family  $\{v_k : k \geq 1\}$  are linearly independent, numerical index of  $\tau$  is  $m$ . It is obvious that  $\tau \in CP_{\phi_0}$  where  $\phi_0$  is the normalized trace. Since  $\{u, v\}'' = M_3(\mathcal{C})$ , we also conclude that  $\tau$  is an ergodic element. Such an ergodic element  $\tau$  is non extremal and  $\tau_\lambda = \lambda\tau + (1 - \lambda)I$  is strong mixing for  $0 < \lambda < 1$ .  $\tau_\lambda$  with  $m = 2$  ( an element with numerical index 3 since  $I$  is not an element in their linear spans of  $v_k : 1 \leq k \leq m$ ) is not cocycle conjugate to the extremal element discussed in Example 2.11 [LS], which admits strong mixing property. ■

### 3 Extremal elements in $CP_\phi$ and associated operator spaces :

Let  $\tau(x) = \sum_{1 \leq k \leq d} v_k x v_k^*$  and  $\tau'(x) = \sum_{1 \leq k \leq d} v'_k x v'_k{}^*$  be two extremal elements in  $CP_\phi$  with numerical index equal to  $d$ . If we have  $uv_k u^* = w \beta_g(v'_k)$  for all  $1 \leq k \leq d$  for some unitary  $u : \mathcal{K} \rightarrow \mathcal{K}'$  such that  $u\Omega = \Omega'$  and  $w : \mathcal{K}' \rightarrow \mathcal{K}'$  and  $g = ((g_j^i)) \in U_d(C)$ . Then we have

$$uv_i v_j^* u^* = w \beta_g(v'_i (v'_j)^*) w^*.$$

Taking trace on both side we get  $g((\phi(v_i v_j^*))) g^* = ((\phi(v'_i (v'_j)^*)))$  Further  $\lambda_j^i v_i v_j^* \Omega \rightarrow \lambda_j^i \beta_g(v'_i (v'_j)^*) \Omega'$  is an inner product preserving map.

Conversely we can fix an element  $g \in U_d(C)$  so that  $g((\phi(v_i v_j^*))) g^* = ((\phi(v'_i (v'_j)^*)))$  if the matrices are having same spectrum. Finer question that we ask now: Does  $\lambda_j^i v_i v_j^* \Omega \rightarrow \lambda_j^i \beta_g(v'_i (v'_j)^*) \Omega'$  is a well defined map preserving inner product so that we can have an extension to an unitary operator  $u_0$  so that

$$u_0 v_i v_j^* u_0^* = \beta_g(v'_i (v'_j)^*).$$

Once that is achieved we may set unitary operator  $w_0 : \mathcal{K}' \rightarrow \mathcal{K}'$  defined by  $w_0 : v_j u_0^* f \rightarrow \beta_g((v'_j)^*) f$  for all  $f \in \mathcal{K}'$ . In such a case we get  $w_0 v_j^* u_0^* = \beta_g(v_j^*)$  for all  $1 \leq j \leq d$ .

The map  $\mathcal{I}_g : \lambda_j^i v_i v_j^* \oplus v_j^* v_i \rightarrow \lambda_j^i \beta_g(v'_i (v'_j)^*) \oplus (v'_j)^* v'_i$  is well defined and invertible for extremal  $\tau$  and  $\tau'$  in  $CP_\phi$  and any fix  $g \in U_d(C)$ .

Given an element  $\tau(x) = \sum_{1 \leq k \leq d} v_k x v_k^*$  with numerical index  $d$ , we set basic data matrices  $D_\phi = ((\phi(v_i v_j^*)))$  associated with a state  $\phi$  is a state on  $\mathcal{A}$ . We denote by  $\mathcal{A}_*$  the set of states on  $\mathcal{A}$ . For an unitary element in  $\mathcal{A}$ , we set  $\mathcal{A}_*^u = \{\phi \in \mathcal{A}_* : \phi(uxu^*) = \phi(x)\}$  and consider the set of data matrices  $D_u = \{D_\phi : \phi \in \mathcal{A}_*^u\}$ . We say two elements  $\tau$  and  $\eta$  of same numerical indices have conjugate data if there exists an unitary  $u \in \mathcal{A}$  and  $g \in U_d(C)$  such that  $((\phi(v_i v_j^*))) = g((\phi(v'_i v'_j{}^*))) g^*$  for all  $\phi \in \mathcal{A}_*^u$ .

In such a case we will use symbol  $\tau \equiv^{u,g} \tau'$ . It is fairly obvious that two unitary cocycle conjugate elements  $\tau$  and  $\tau'$  with equal numerical index have conjugate data. Similarly statement is also valid for anti-unitary element  $u$  and basic data.

Question that we now face: how about the converse? We first deal with unitary situation. If  $u = I$ , then conjugate relation says that  $\phi(v_i v_j^* - \beta_g(v'_i) \beta_g(v'_j{}^*)) = 0$  for all  $i, j$  and  $\phi \in \mathcal{A}_*$ . Thus  $\tau$  and  $\tau'$  are conjugate i.e.  $v_i w = \beta_g(v_i)$  for  $1 \leq i \leq d$  where  $w$  is an unitary operator. In general we claim that the following two statements are equivalent:

(a) For any matrix  $((\lambda_j^i))$

$$\sum \lambda_j^i v_i v_j^* > 0 \text{ then } \sum \lambda_j^i \beta_g(v'_i) \beta_g(v'_j{}^*) > 0 \quad (3.1)$$

(b) For any matrix  $((\lambda_j^i))$

$$\sum \lambda_j^i v_i v_j^* \geq 0 \text{ then } \sum \lambda_j^i \beta_g(v'_i) \beta_g(v'_j{}^*) \geq 0 \quad (3.2)$$

as  $\sum_i v_i v_i^* = 1 = \sum_i v'_i v'_i{}^*$ . Linear independence of elements  $\{v_i v_j^* : 1 \leq i, j \leq d\}$  will ensure that such an element  $\lambda$  is self-adjoint. Further we also note by scaling property that equivalent statements (3.1) and (3.2) are valid if same holds for all elements  $\lambda \in S_h^1 = \{\lambda \in M_d(C) : \lambda = \lambda^*, ||\lambda|| = 1\}$ , which is compact.

The proof for converse is far from being straight forward. We prove that in the following by splitting into a sequence lemmas:

**LEMMA 3.1:** The set of extremal elements  $\mathcal{L}_d^e$  ( which could be empty ) in  $\mathcal{L}_d$  forms an open subset of  $\bar{\mathcal{L}}_d$  with respect to subspace topology.

**PROOF:** We will show complement is closed. Given a sequence of non-extremal elements  $\tau_n \in \mathcal{L}_d$ , we find unit elements  $\lambda_n \in M_n(C)$  such that  $\sum \lambda_j^i(n) v_i(n) v_j^*(n) = 0$ . Now we use compactness of unit ball of  $M_d(C)$  to extract a convergent subsequence with limiting value say  $\lambda$  with norm 1. Thus  $\lambda_j^i v_i v_j^* = 0$  if  $\tau_n(x) \rightarrow \tau(x) = \sum_k v_k x v_k^*$  in  $\mathcal{L}_d$ . Thus  $\tau$  is also non-extremal. ■

For a fix  $\tau \in \mathcal{L}_d^e$ , we consider now the set  $E_\tau$  of elements in the closer of  $\mathcal{L}_d^e$  with conjugate data of  $\tau$  i.e.

$$E_\tau = \{\tau' \in \bar{\mathcal{L}}_d^e : \tau' \equiv^{g,u} \tau, \text{ for some } g \in U_d(C), u \in \mathcal{A} \text{ unitaries} \}$$

We also consider the subset  $E'_\tau$  defined by

$$E'_\tau = \{\tau' \in \bar{\mathcal{L}}_d^e : \tau' \equiv^{g,u} \tau, \text{ for some } g \in U_d(C), u \in \mathcal{A} \text{ unitaries so that (3.1) holds} \}$$

**LEMMA 3.2:**  $E'_\tau$  is closed and as well as open as a subset of  $E_\tau$  with respect to subspace topology.

**PROOF:** That  $E'_\tau$  is a closed subset of  $E_\tau$  follows by (3.2) once we use compactness of  $U_d(C)$  and  $U_n(C)$ . Now we claim that  $E'_\tau$  is also an open set in  $E_\tau$ . We fix  $\tau' \in E'_\tau$ . Given  $\lambda \in S_h^1$ , we can find an open neighborhood  $\mathcal{O}_\lambda$  of  $\lambda$  and  $\mathcal{O}_{\tau',\lambda}$  of  $\tau'$  such that (3.1) is valid for all  $\lambda' \in \mathcal{O}_\lambda$  and  $\tau'' \in \mathcal{O}_{\tau',\lambda}$ . Suppose not. Then we would have got sequence of elements  $\tau_n \rightarrow \tau \in E_\tau$ ,  $g_n \rightarrow g \in U_d(C)$ ,  $\lambda_n \rightarrow \lambda \in S_h^1$  and states  $\psi_n$  so that  $\psi_n(\sum_{i,j} \lambda_j^i(n) \beta_{g_n}(v'_i(n)) \beta_{g_n}(v'^*_j(n))) \leq 0$  for all  $n \geq 1$  where  $\sum \lambda_j^i(n) v_i v_j^* > 0$ . Once more we use compactness to extract a sub-sequence so that we get  $\psi(\sum_{i,j} \lambda_j^i \beta_g(v'_i) \beta_g(v'^*_j)) \leq 0$  for a state  $\psi$ . This brings a contradiction as  $\sum_{i,j} \lambda_j^i v_i v_j^* > 0$  and  $\lambda_j^i \beta_g(v'_i) \beta_g(v'^*_j) > 0$ .

Now we look for a finite sub-cover  $\cup \mathcal{O}_{\lambda_k}$  of  $S_h^1$  and take open set  $\mathcal{O} = \bigcap \mathcal{O}_{\tau',\lambda_k}$  which is a neighborhood of  $\tau'$  so that for all  $\tau'' \in \mathcal{O}$ , (3.1) is valid. This shows that  $E'_\tau$  is open in  $E_\tau$ . ■

**LEMMA 3.3:** For an element  $\eta \in E_\tau$ , there exists an open set  $\mathcal{O}$  of  $I \in \mathcal{A}$  so that if  $\eta' \in \bar{\mathcal{L}}_d$  and  $\eta \equiv^{g,u} \eta'$  with some element  $g \in U_d(C)$  and  $u \in \mathcal{O}$ , then  $\eta'$  is an element in the connected components of  $\eta$  in  $E_\tau$ .

**PROOF:** Suppose not. Then we will have  $\eta \equiv^{g_n, u_n} \eta'_n$  for some sequence of elements  $g_n \in U_d(C)$  and unitaries  $u_n \in \mathcal{A}$  so that  $u_n \rightarrow I$  as  $n \rightarrow \infty$  and  $\eta'_n$  is not in the same connected component of  $\tau$ . Taking a sub-sequence if needed we can get  $g_n \rightarrow g$  for

some  $g \in U_d(C)$  and  $\eta'_n \rightarrow \eta'$  as  $n \rightarrow \infty$ . We claim that  $\eta'$  belongs to the connected components of  $\tau$  in  $E_\tau$ . For any state  $\psi$  we take  $\psi_n = \psi E_n$  where  $E_n : \mathcal{A} \rightarrow \mathcal{A}$  is the norm one projection on the algebra of  $\{u_n\}''$ . Then  $\psi_n$  is  $\alpha_n(x) = u_n x u_n^*$  invariant state and  $\psi_n(\lambda_j^i v_i v_j^*) = \psi_n(\lambda_j^i \beta_{g_n}(v_i'(n)) \beta_{g_n}(v_j'^*(n)))$  for all  $n \geq 1$ . Thus taking limit  $n \rightarrow \infty$ , we conclude that  $\eta' \equiv_{I,g} \eta$ . Thus there exists a unitary matrix extending the map defined by  $u : v_i^* f \rightarrow v_i'^* f$  for  $f \in C^n$  and so  $\eta'(x) = \eta(u^* x u)$  for some  $u \in U_n(C)$ . Further  $U_n(\mathcal{C})$  being path connected, we conclude that  $\eta'$  belongs to the same closed-open connected subset of  $E_\tau$ . This brings a contradiction as each  $\eta'_n$  is not an element in the connected closed-open neighborhood where  $\eta'$  belongs to. ■

**PROPOSITION 3.4:**  $E_\tau = E'_\tau$ . Further  $E_\tau$  is path connected.

**PROOF:** First we need a little more general statement then what we have proved in Lemma 3.3. We claim that for any element  $\eta \in E'_\tau$  in the connected component of  $\tau$  we can choose an open subset  $\mathcal{O}_\eta$  in  $E'_\tau$  such that Lemma 3.3 holds true for any element in that open set  $\mathcal{O}_\eta$  uniformly for an open set  $\mathcal{O}_I^\eta$  of  $I \in \mathcal{A}$ . To that end we reset notations  $\tau_0$  for  $\tau$ ,  $E_0 = E_{\tau_0}$  and  $E'_0 = E'_{\tau_0}$ . Further let  $E'_0(\tau_0)$  be the connected component  $\tau_0$  in  $E'_0$ . We claim that given an element  $\eta \in E'_0(\tau_0)$ , there exists an open neighborhood  $\mathcal{O}_\eta$  of  $\eta$ , a subset of  $E'_0(\tau_0)$  and an open neighborhood  $\mathcal{O}_I^\eta$  of  $I \in U_n(C)$  so that the following hold:

If for any element  $\tau \in \mathcal{O}_\eta$  with  $\tau \equiv_{g,u} \tau'$  with some  $g \in U_d(C)$  and  $u \in \mathcal{O}_I^\eta$  then (3.1) hold with  $\tau, \tau'$  and  $\tau, \tau'$  are elements in  $E'_0(\tau_0)$ .

Suppose not. Then by Lemma 3.2 we find a sequence of elements  $\tau_n, \tau'_n$  satisfying  $\tau_n \equiv_{u_n, g_n} \tau'_n$  for some  $g_n \in U_d(C)$  and unitary elements  $u_n \in \mathcal{A}$  so that  $\tau_n \rightarrow \eta$ ,  $u_n \rightarrow I$  and  $\tau'_n$  does not belong to the same connected components of  $\tau_0$  in  $E'_0$ . Once more extracting a sub-sequence if needed, we ensure that  $\tau'_n \rightarrow \tau'$  for some  $\tau' \in E_0$ . Now we follow the proof of Lemma 3.3 to conclude once again that  $\eta \equiv_{I,g} \tau'$  to bring a contradiction.

Now we choose a finite open cover  $\bigcup \mathcal{O}_{\eta_k}$  for  $E'_0(\tau_0)$  which is compact and choose

open set  $\mathcal{O}_I = \bigcap \mathcal{O}_I^{\eta_k}$ , neighborhood of  $I$  in  $U_n(C)$ . Thus we find an open neighborhood  $\mathcal{O}_I$  of  $I \in \mathcal{A}$  which satisfies the following:

For any element  $\tau' \in E_0$  if  $\tau' \equiv_{g,u} \tau_0$  for some  $g \in U_d(C)$  and  $u \in \mathcal{O}_I$  then  $\tau' \in E'_0(\tau_0)$ .

Let  $E_0(\tau_0)$  be the connected component in  $E_0$  with  $\tau_0$ . We claim that for any elements  $\tau' \in E_0(\tau_0)$  (3.1) is true with  $\tau = \tau_0$ . For a quick proof we choose a path connecting  $\tau_0$  and  $\tau'$  and fix a finite open sub-cover  $\mathcal{O}_{\tau_k}$  where each  $\tau_k$  are extremal elements ( by our construction extremal elements are dense in  $E_0(\tau_0)$  ) and for any other extremal element  $\tau'_k$  in  $\mathcal{O}_{\tau_k}$  (3.1) holds with  $\tau = \tau_k$  and  $\tau' = \tau'_k$ . Note that property (3.1) is transitive and so  $\tau'$  also satisfies (3.1) with  $\tau = \tau_0$  in (3.1). Note that we are not assuming  $\tau'$  to be an extremal element. This shows that  $E_0(\tau_0) = E'_0(\tau_0)$ .

So we left to show that  $E_0$  is indeed connected. Suppose not. Then we set

$$U = \{u \in U_n(C) \subseteq \mathcal{A} : \tau' \notin E_0(\tau_0) : \tau' \equiv_{(u,g)} \tau \text{ for some } g \in U_d(C), \tau \in E_0(\tau_0)\}$$

which is non-empty and neither equal to  $U_n(C)$ .

We claim that  $U$  is both open and closed to bring a contradiction. That  $U$  is closed follows as  $E_0(\tau_0), E_0(\tau_0)^c$  are also closed once we use compactness of  $CP$ ,  $U_d(C)$  and  $U_n(C) \subseteq \mathcal{A}$ . To prove  $U$  is also open we fix an element  $u \in U$ . Let  $\tau$  be an extremal element in  $E_0(\tau_0)^c$  such that  $\tau_0 \equiv_{(g,u)} \tau$ . Since  $E_0(\tau_0)^c$  is an open set, we can find applying Lemma 3.3 an open neighborhood  $\mathcal{O}_I$  of  $I$  in  $U_n(C)$  so that the relation  $\tau \equiv_{(v,h)} \tau'$  for some  $h \in U_d(C)$  and  $v \in \mathcal{O}_I \subseteq U_n(C)$  ensures that  $\tau' \in \mathcal{O}_\tau \subseteq E_0(\tau_0)^c$ . Thus  $\{vu : v \in \mathcal{O}_I\} \subseteq U^c$  as  $\tau_0^v \equiv_{(hg,vu)} \tau'^{u*}$  where  $\tau_0^v(x) = v\tau_0(x)v^*$  and  $\tau'^{u*}(x) = u^*\tau'(x)u$ . This shows that  $U$  is an open set. This brings a contradiction as  $U_n(C)$  is connected. So  $E_0(\tau_0) = E_0$  and  $E_0$  is path connected. ■

**THEOREM 3.5:** Let  $\tau, \tau'$  be two extremal elements in the set of unital completely positive map on  $\mathcal{A} = M_n(C)$  with equal numerical index  $d$  and  $\mathcal{S}, \mathcal{S}'$  be the associated



operator systems in  $\mathcal{A}$ . Then there exists an order isomorphic map

$$\mathcal{I}_{g,u} : \lambda_j^i v_i v_j^* \rightarrow \lambda_j^i \beta_g(v_i') \beta_g(v_j'^*)$$

for some  $g \in U_d(C)$  if  $\tau, \tau'$  are having unitary conjugate basic data i.e.  $\tau \equiv_{u,g} \tau'$  where  $u$  is an unitary element in  $\mathcal{A}$ . Further  $\phi(I_g(x)) = \phi(x)$  for all  $x \in \mathcal{S}$  and all states  $\phi \in \mathcal{A}_*^u = \{\phi \in \mathcal{A}_*^1 : \phi(uxu^*) = \phi(x), x \in \mathcal{A}\}$ . Similar statements also hold for anti-unitary conjugate data.

**PROOF:** Proof of the first part follows from Proposition 3.4 by applying twice. For anti-unitary conjugate data, we consider the GNS space  $(\mathcal{H}, \pi, \Omega)$  associated with tracial state  $\phi$  on  $\mathcal{A}$  and set elements  $v_k'' = \mathcal{J}v_k' \mathcal{J}$  in the commutant which once identified with  $M_n(C)$  we get a unital CP map  $\tau''(x) = \mathcal{J}\tau'(\mathcal{J}x\mathcal{J})\mathcal{J}$  where  $\mathcal{J}$  is Tomita's conjugation operator and  $M_n(C)$  is identified with the commutant of  $\pi(\mathcal{A})$  in the GNS space.

We claim if  $g_0$  is the intertwining anti-unitary operator of basic data matrices  $((\psi(v_i v_j^*)))$  and  $((\psi(v_i' v_j'^*)))$  where  $\psi$  is a state on  $\mathcal{A}$ , then  $g_0 \mathcal{J}_0$  is the unitary operator intertwines basic data matrices of  $\tau$  and  $\tau''$ , where  $\mathcal{J}_0((z_k)) = ((\bar{z}_k))$  is the complex conjugation on  $C^n$ . This follows as  $\psi(\mathcal{J}_\psi x \mathcal{J}_\psi) = \psi(x^*)$  for any faithful state  $\psi$  where  $\mathcal{J}_\psi$  is Tomita's conjugate operator associated with  $\psi$  and  $\psi(x) = \langle \zeta_\psi, x \zeta_\psi \rangle$  with  $\zeta_\psi$ , an unit vector in the positive self dual cone which is closer of the linear space  $\{x \mathcal{J} x \mathcal{J} : x \in \mathcal{A}\}$ . However  $\mathcal{J}_\psi = \mathcal{J}$  [BR 1] and thus we have  $\psi(\mathcal{J} x \mathcal{J}) = \psi(x^*)$ . That we can drop the assumption faithfulness follows by a limiting argument by considering faithful states  $\psi_\lambda = (1 - \lambda)\phi + \lambda\psi$  for  $\lambda \in [0, 1)$ . Thus in particular  $\mathcal{J}_0 \psi(v_i' v_i'^*) \mathcal{J}_0 = \psi(\mathcal{J} v_i' v_i'^* \mathcal{J})$ . Thus we can apply first part by taking  $g = g_0 \mathcal{J}_0$ . ■

**COROLLARY 3.6:** Let  $\tau, \tau'$  be as in Theorem 3.5. If  $\mathcal{S}$  and  $\mathcal{S}'$  are  $C^*$ -sub-algebras of  $\mathcal{A}$ , then the map  $\mathcal{I}_g : \mathcal{S} \rightarrow \mathcal{S}'$  for some  $g \in U_d(C)$  is an isomorphism if and only if  $\tau \equiv_{u,g} \tau'$  for some unitary operators  $u, g$ . Further  $\phi(\mathcal{I}_g(x)) = \phi(x)$  for all  $x \in \mathcal{S}$  and  $\phi \in \mathcal{A}_*^u$ . Similar statement holds for anti-conjugate basic data.

**PROOF:** Any order-isomorphism between two unital  $C^*$ -algebras admits Jordan

decomposition i.e. there exists a projection  $E$  in the center of  $\mathcal{S}'$  such that  $x \rightarrow \mathcal{I}_g(x)E$  is a morphism and  $x \rightarrow \mathcal{I}_g(x)(I - E)$  is an anti-morphism. If  $E \neq I$  then the later case the map  $\mathcal{I}_{g,u}$

$$\lambda_j^i v_i v_j^* \rightarrow \lambda_j'^i v_i' v_j'^* (1 - E)$$

induces an anti-morphism map

$$\lambda_j^i \rightarrow \lambda_j'^i$$

but  $g$  is assumed to be unitary. Thus  $E = I$  and  $\mathcal{I}_g(x) = wxw^*$  for some unitary  $w$ . For anti-unitary basic data, we adopt the modified argument as in Theorem 3.5. ■

**THEOREM 3.7:** Let  $\tau, \tau'$  be two extremal elements in the set of trace preserving unital completely positive maps on  $\mathcal{A} = M_n(C)$  with equal numerical index  $d$  and  $\mathcal{S}, \mathcal{S}'$  be the associated operators systems in  $\mathcal{A} \oplus \mathcal{A}$ . Then there exists an order isomorphic map

$$\mathcal{I}_{g,u} : \lambda_j^i v_i v_j^* \oplus v_j^* v_i \rightarrow \lambda_j^i \beta_g(v_i') \beta_g(v_j'^*) \oplus \beta_g(v_j'^*) \beta_g(v_i')$$

for some  $g \in U_d(C)$  if  $\tau, \tau'$  are having unitary conjugate basis pair data i.e.

$$((\phi(v_i v_j^* \oplus v_j v_i^*)))$$

and

$$((\phi(v_i' v_j'^* \oplus v_j' v_i'^*)))$$

are intertwined by unitary matrix  $g$  for all  $\phi \in (\mathcal{A} \otimes M_2(C))^u_* = \{\phi \in (\mathcal{A} \otimes M_2(C))^* : \phi(x) = \phi(uxu^*), \forall x \in \mathcal{A} \otimes M_2(C)\}$  for some unitary element  $u \in \mathcal{A} \otimes M_2(C)$ . Similar statements also holds for anti-unitary conjugate data.

**PROOF:** Proofs goes along the same line as that of Theorem 3.5. ■

**COROLLARY 3.8:** Let  $\tau, \tau'$  be as in Theorem 3.7. If  $\mathcal{S}$  and  $\mathcal{S}'$  are  $C^*$ -sub-algebras of  $\mathcal{A}$ , then the order isomorphism map  $\mathcal{I}_{g,u} : \mathcal{S} \rightarrow \mathcal{S}'$  for some  $g \in U_d(C)$  defined in Theorem 3.7 is an isomorphism and  $I_g(x) = wxw^*$  for all  $x \in \mathcal{A} \otimes M_2(C)$  for some

unitary operator  $w \in \mathcal{A} \otimes M_2(\mathbb{C})$ . Similar statement holds for anti-conjugate basic data.

**PROOF:** Proof goes along the same line as that of Corollary 3.6. ■

## 4 Operator spaces of extremal elements in $CP$ and $CP_\phi$ :

Results in the previous section give rise to question when such an order isomorphism  $\mathcal{I}_g : \mathcal{S} \rightarrow \mathcal{S}''$  induces an order isomorphism to their minimal  $C^*$ -algebras and gets implemented by an unitary operator. We start with an interesting observation.

**PROPOSITION 4.1:** Let  $\mathcal{S}$  be an operator subspace of an unital  $C^*$ -algebra  $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$  such that  $\mathcal{S}$  contains unit element and the closed  $C^*$  algebra generated by  $\mathcal{S}$  be equal to  $\mathcal{A}$ . Let  $\tau$  be an unital CP map on  $\mathcal{A}$  with a faithful invariant state extending the inclusion map  $I : \mathcal{S} \rightarrow \mathcal{A}$ . Then  $\tau$  is the identify map on  $\mathcal{A}$ .

**PROOF:** It is fairly well known that the set  $\mathcal{N} = \{x \in \mathcal{A} : \tau(x) = x\}$  is a  $*$ -algebra for an unital CP map with a faithful invariant state. Proof goes as follows. By Kadison inequality we have  $\tau(x^*x) \geq \tau(x^*)\tau(x)$  for all  $x \in \mathcal{A}$  and if equality hold for  $x$  then we also have  $\tau(x^*y) = \tau(x^*)\tau(y)$  for all  $y \in \mathcal{A}$ . Now we use faithfulness of the invariant state to show first that  $x^*x$  in  $\mathcal{N}$  whenever  $x \in \mathcal{N}$  and then  $x^*y \in \mathcal{N}$  when  $x, y \in \mathcal{N}$ .  $\tau$  being an extension of identity map on  $\mathcal{S}$ , it contains  $\mathcal{S}$ . Thus  $\mathcal{N}$  also contains  $*$ -algebra generated by  $\mathcal{S}$ .  $\tau$  being continuous, we conclude that  $\mathcal{N} = \mathcal{A}$  i.e.  $\tau$  is the identity map on  $\mathcal{A}$ . ■

Now we recall essential steps in Hann-Banach-Arveson's extension theorem [Pa, Chapter 6]. We set one to one correspondence between the set of completely positive maps  $\tau : \mathcal{S} \rightarrow M_n(\mathbb{C})$  and positive functional  $s_\tau : M_n(\mathcal{S}) \rightarrow \mathbb{C}$  defined by

$$s_\tau((x_j^i)) = \frac{1}{n} \sum_{1 \leq i, j \leq d} \tau(x_j^i)_j^i$$

and

$$\tau_s(x)_j^i = ns(x \otimes |e_i \rangle \langle e_j|)$$

where  $e_i$  is a orthonormal basis for  $C^n$ . Note that  $s_\tau(x \otimes I_n) = tr(\tau(x))$  for all  $x \in \mathcal{S}$  where  $tr$  is the normalized trace on  $M_n(C)$  and further

- (a)  $\tau$  is unital if and only if  $s_\tau$  is a state on the operator space  $M_n(\mathcal{S})$ ;
- (b) If  $\mathcal{A}$  is with a normalized trace, then  $\tau$  is trace preserving if and only if  $s_\tau(x \otimes I_n) = tr(x)$ ; where  $tr$  is a normalized trace on  $\mathcal{A}$ .

In order to deal with a faithful state  $\phi$  on  $M_n(C)$  we fix an orthonormal basis  $e_i$  and  $\{\lambda_i \neq 0$  such that  $\phi_0(x) = \sum_i |\lambda_i|^2 \langle e_i, x e_i \rangle$  for all  $x \in M_n(C)$  and consider the non-negative matrix  $\lambda = ((\bar{\lambda}_i \lambda_j))$  and reset

$$s_{\tau, \lambda}((x_j^i)) = s_\tau(((\lambda_j^i)) \circ ((x_j^i)))$$

$$\tau_s(x)_j^i = \frac{1}{\lambda_j^i} s_{\tau, \lambda}(x \otimes |e_i \rangle \langle e_j|)$$

where  $\circ$  denotes Schur product. Since Schur product takes a non-negative element to another non-negative element on  $M_n(\mathcal{S})$ , we inherits all the property of the correspondence between  $\tau$  and  $s_\tau$  with a modification  $s_{\tau, \lambda}(x \otimes I_n) = \phi_0(\tau(x))$  for all  $x \in \mathcal{A}$ . Thus (b) is now modified as

- (b') If  $\phi$  is a state on  $\mathcal{A}$  then  $\phi_0(\tau(x)) = \phi(x)$  for all  $x \in \mathcal{S}$  if and only if  $s_{\tau, \lambda}(x \otimes I_n) = \phi(x)$  for all  $x \in \mathcal{S}$ .

**THEOREM 4.2:** Let  $\mathcal{S}$  be an operator system in an unital  $C^*$  algebra  $\mathcal{A}$  with a normalized trace and  $\tau : \mathcal{S} \rightarrow M_n(C)$  be a unital completely positive trace preserving map. Then  $\tau$  has a unital trace preserving completely positive extension  $\tau : \mathcal{A} \rightarrow M_n(C)$ . Same holds true with a state on  $\mathcal{A}$  also i.e. If  $\phi_0(\tau(x)) = \phi(x)$  for all  $x \in \mathcal{S}$  for a state  $\phi$  on  $\mathcal{A}$  and  $\phi_0$  a faithful state on  $M_n(C)$  then there exists an Arveson's extension on  $\tau : \mathcal{A} \rightarrow M_n(C)$  so that  $\phi_0(\tau(x)) = \phi(x)$  for all  $x \in \mathcal{A}$ .

**PROOF:** We consider the linear function  $\hat{s}$  on the operator space the linear span of  $M_n(\mathcal{S})$  and  $\mathcal{A} \otimes I_n$  defined by  $\hat{s}(((x_j^i)) + x \otimes I_n) = s_\tau(((x_j^i))) + tr(x)$  where  $x_j^i \in \mathcal{S}$

and  $x \in \mathcal{A}$ . That it is well defined follows as  $s_\tau(x \otimes I) = \text{tr}(x)$  for  $x \in \mathcal{S}$ . We claim that  $\hat{s}$  is contractive. The linear functional being on an operator system, contractive property is equivalent to positivity of  $\hat{s}$ . For any element  $Y = ((x_j^i)) + x \otimes I_n \geq 0$ , with representing element  $x_j^i \in \mathcal{S}$  and  $x \in \mathcal{A}$ , we assume without loss of generality that  $x$  and  $((x_j^i))$  are self-adjoint elements. We consider the maximal set  $\mathcal{M}$  of self-adjoint elements  $x \in \mathcal{A}$  for which whenever  $((x_j^i)) + x \otimes I_n \geq 0$  for  $x_j^i \in \mathcal{S}$  then  $\hat{s}((x_j^i) + x \otimes I_n) \geq 0$ . Certainly  $I \in \mathcal{M}$  and if  $x \in \mathcal{M}$  then  $x - I \in \mathcal{M}$  and also  $\lambda x \in \mathcal{M}$  for all  $\lambda > 0$  if  $x \in \mathcal{M}$ .

We aim to prove  $\mathcal{M} = \mathcal{A}_h$ , self-adjoint elements in  $\mathcal{A}$ . Suppose not. Then for any neighborhood  $\mathcal{O}$  of  $I$ , there exists an element  $x(\neq \|x\|I) \in \mathcal{O}$  such that  $x \notin \mathcal{M}$ . Let  $\mathcal{O}_\epsilon$  be a sequence of open sets that shrinks to the set consist of only identity element as  $\epsilon \rightarrow 1$ . Further we can choose  $x_\epsilon \rightarrow I$  as  $\epsilon \rightarrow 1$  with additional property:

$$0 \leq x_\epsilon \leq \|x_\epsilon\|$$

and

$$\text{tr}(x_\epsilon) = \epsilon \|x_\epsilon\|.$$

Since  $\epsilon x_\epsilon \notin \mathcal{M}$ , we find an element  $Y_\epsilon = (x_j^i(\epsilon)) + \epsilon x_\epsilon \otimes I_n \geq 0$  but  $\hat{s}(Y_\epsilon) < 0$ . However by our choice we also check that

$$((x_j^i(\epsilon))) \geq -\epsilon \|x_\epsilon\| I \otimes I_n$$

and so by positivity of  $s$  we have  $s((x_j^i(\epsilon))) \geq -\epsilon \|x_\epsilon\| = -\text{tr}(x_\epsilon)$  i.e.  $\hat{s}(Y_\epsilon) \geq 0$ . This brings a contradiction. Thus  $\mathcal{M} = \mathcal{A}_h$ .

$\hat{s}$  is a contractive unital map on an operator space. So by Krein's theorem we can extend  $\hat{s}$  to  $M_n(\mathcal{A})$  as a state. Thus we get a trace preserving unital extension of  $\tau : \mathcal{S} \rightarrow M_n(C)$  to  $\tau : \mathcal{A} \rightarrow M_n(C)$  by the above correspondence discussed preceding the statement of this theorem.

We need to include very little modification of the above argument with  $s_{\tau,\lambda}$  replacing the role of  $s_\tau$  in order to include more general state  $\phi$  on  $\mathcal{A}$ . ■

Now onwards for our application we assume all the  $C^*$ -algebras in the following are finite dimensional. A one to one and onto map  $\mathcal{I} : \mathcal{S} \rightarrow \mathcal{S}'$  between two operator space is called complete order isomorphism if both  $\mathcal{I}$  and it's inverse are completely positive i.e.  $\mathcal{I} \otimes I_n(M_n(\mathcal{S})_+) \subseteq M_n(\mathcal{S}')_+$  and also  $\mathcal{I}^{-1} \otimes I_n(M_n(\mathcal{S}')_+) \subseteq M_n(\mathcal{S})_+$  for all  $n \geq 1$ .

**THEOREM 4.3:** Let  $\mathcal{I} : \mathcal{S} \rightarrow \mathcal{S}'$  be an unital complete order isomorphism where  $\mathcal{S}$  and  $\mathcal{S}'$  are unital operator  $*$ -subspaces of  $C^*$  algebras  $\mathcal{A}$  and  $\mathcal{A}'$  respectively. Further assume that each  $C^*$ -algebras admits a normalized trace and complete order isomorphism preserves their traces on  $\mathcal{S}$  and  $\mathcal{S}'$ . Then  $\mathcal{I}$  has a complete order isomorphic extension to  $\mathcal{I} : \mathcal{B} \rightarrow \mathcal{B}'$  preserving traces where  $\mathcal{B}$  and  $\mathcal{B}'$  are the minimal unital  $C^*$  subalgebras of  $\mathcal{A}$  and  $\mathcal{A}'$  containing  $\mathcal{S}$  and  $\mathcal{S}'$  respectively.

**PROOF:** Without loss of generality we assume that  $\mathcal{A}$  and  $\mathcal{A}'$  are generated by  $\mathcal{S}$  and  $\mathcal{S}'$  respectively. Let  $\tau$  and  $\eta$  be Arveson's extension of the map  $\mathcal{I}$  and it's inverse preserving normalized traces respectively. Then  $\eta \circ \tau$  is an unital map and an extension of the inclusion map of  $\mathcal{S}$  in  $\mathcal{A}$  preserving trace and thus by Proposition 4.2  $\eta \circ \tau$  is trivial on  $\mathcal{A}$ . Same is also true for  $\tau \circ \eta$ . Thus  $\eta$  is the inverse of  $\tau$ . This completes the proof. ■

**THEOREM 4.4:** Let  $\tau(x) = \sum_{1 \leq k \leq d} v_k x v_k^*$  and  $\tau'(x) = \sum_{1 \leq k \leq d} v'_k x (v'_k)^*$  be two extremal elements in the set of unital  $CP$  maps and their numerical index are equal to  $d \geq 1$ . Then  $\tau$  and  $\tau'$  are having unitary conjugate data if and only if the map  $I_{g,u} : (\mathcal{S}, \mathcal{C}_n) \rightarrow (\mathcal{S}', \mathcal{C}'_n)$  defined in Theorem 3.9 is a completely positive order isomorphic map. Further in such a  $I_{g,u}(x) = w x w^*$  for an unitary operator  $w \in \mathcal{A}$ . Similar statement also holds true for anti-unitary conjugate data.

**PROOF:** That the map is positive for each  $n \geq 1$  will follow along the line of Lemma 3.2, Lemma 3.3 and Lemma 3.4 with obvious modification in the definition of  $E'_0$  for the matrix order  $C_n^2, C_n'^2$ . We omit the details as it follows with verbatim changes. The last part follows by Theorem 4.3 as  $I_{g,u}$  is trace preserving. ■

**THEOREM 4.5:** Let  $\tau(x) = \sum_{1 \leq k \leq d} v_k x v_k^*$  and  $\tau'(x) = \sum_{1 \leq k \leq d} v'_k x (v'_k)^*$  be two extremal elements in the set of unital trace preserving completely positive maps  $CP_\phi$  with numerical index equal to  $d \geq 1$ . The unit preserving order isomorphism map defined in Theorem 3.7  $\mathcal{I}_{(g,u)} : (\mathcal{S}, \mathcal{C}_n) \rightarrow (\mathcal{S}', \mathcal{C}'_n)$  for unitary conjugate data is a complete order isomorphism. Further any two such extremal elements are cocycle conjugate by unitary operators. Similar statements also holds for anti-unitary basis data.

**PROOF:** Proof goes along the same line of that of Theorem 4.4 to get an unitary element  $w \in \mathcal{A} \otimes M_2(C)$ . We claim that  $w$  commutes with  $I_d \oplus 0$ . Since  $\tau'$  and  $\tau$  can be connected by a path with extremal elements in  $E_\tau$  as set of extremal elements are open and dense and so such a property is true with  $\tau'$  in a neighborhood of  $\tau$ , we conclude it is also true for any  $\tau'$  by transitive property of the equivalence relation. Thus we conclude that  $\tau$  and  $\tau'$  are cocycle conjugate by unitary operators. ■

For a given element  $\tau \in CP_\phi$ , we denote the adjoint element  $\tilde{\tau} \in CP_\phi$  defined by the dual relation  $\phi(x\tau(y)) = \phi(\tilde{\tau}(x)y)$  for all  $x, y \in M_n(C)$ . If  $\tau(x) = \sum_k v_k x v_k^*$  is a representation for  $\tau$  then  $\tilde{\tau}(x) = \sum v_k^* x v_k$  is a representation of  $\tilde{\tau}$ . It is clear that numerical index of  $\tau$  is equal to numerical index of  $\tilde{\tau}$ .

**THEOREM 4.6:** Let  $\tau$  be an extremal element in  $CP_\phi$ . Then  $\tau$  and  $\tilde{\tau}$  are cocycle conjugate by anti-unitary operators. Further  $\tau$  is also an extremal element in the convex set of unital completely positive maps on  $\mathcal{A}$ .

**PROOF:** We consider the automorphism  $\alpha : \mathcal{A} \oplus \mathcal{A} \rightarrow \mathcal{A} \oplus \mathcal{A}$  defined by  $\alpha(X \oplus Y) = Y \oplus X$  extending linearly. We consider a state  $\psi$  that is  $\alpha$  invariant and check that matrices  $((\psi(v_i v_j^* \oplus v_j^* v_i)))$  and  $((\psi(v_i^* v_j \oplus v_j v_i^*)))$  are intertwined by conjugate action i.e.

$$\overline{\psi(v_i v_j^* \oplus v_j^* v_i)} = \psi(v_j v_i^* \oplus v_i^* v_j) = \psi(\alpha(v_j v_i^* \oplus v_i^* v_j)) = \psi(v_i^* v_j \oplus v_j v_i^*)$$

Thus there exists unitary operator  $u \in \mathcal{A} \otimes M_2(C)$  so that  $\mathcal{J}_0 v_i v_j^* \mathcal{J}_0 \oplus \mathcal{J}_0 v_j^* v_i \mathcal{J}_0 = u^*(v_i^* v_j \oplus v_j v_i^*) u$  where  $\mathcal{J}_0$  is conjugation on  $\mathcal{A}$ . Thus we arrive at  $\mathcal{J}_0 v_i v_j^* \mathcal{J}_0 =$

$wv_i^*v_jw^*$  for some unitary  $w \in \mathcal{A}$  as in Theorem 4.5. Let  $\lambda_j^i$  be a matrix such that  $\lambda_j^i v_i v_j^* = 0$ . Then by conjugating with anti-unitary operator  $\mathcal{J}_0 w$ , we conclude that  $\bar{\lambda}_j^i v_i^* v_j = 0$ . Again taking adjoint action, we conclude  $\sum \lambda_j^i v_j^* v_i = 0$ . Now we use Landau-Streater criteria given in Theorem 2.6 to conclude that  $\tau$  is also an extremal element in the set of unital  $CP$  maps. ■

**THEOREM 4.7:** Any extremal element  $\tau \in CP_\phi$  with numerical index  $d \geq 2$  fails to have factorizable property, where we refer [KuM,MuH] for definition of factorizable property.

**PROOF:** It is a simple consequence of Theorem 4.6 once we recall Corollary 2.3 in [MuH]. ■

A more general situation of the present problem would include  $\mathcal{A}$  where  $\mathcal{A}$  is a type-I von-Neumann algebra acting on a separable Hilbert  $\mathcal{H}$  with center completely atomic. If each minimal central element are finite dimensional, then  $\mathcal{A}$  is a direct copies of countably many matrix algebras and trace need not be unique. If we take  $\phi$  to be the restriction of the unique tracial weight on  $\mathcal{B}(\mathcal{H})$ , an automorphism on  $\mathcal{A}$  will preserve  $\phi$ . The set  $CP_\phi$  need not be compact in the topology induced by the weak\* topology as in the classical case unless  $\mathcal{H}$  is finite dimensional. A natural question that we leave here open along the line of G. Birkhoff's problem: What are it's extreme points and can we write an element in  $CP_\phi$  as a convex combination of it's extremal elements?

## 5 Some computation:

So far we did not comment on existence of extremal element in  $CP$  or  $CP_\phi$  with a given numerical index and matrix data. We start here with the simplest case namely  $\mathcal{A} = M_2(C)$  and leave it for a future direction of work for higher dimension. In case  $CP_\phi$ , we have nothing to say as it is well known all it's extremal elements are automorphisms. For  $CP$  we will show computation which leads to an extremal



element with index  $d = 2$  with a given data matrix. For an given element  $\tau(x) = v_1 x v_1^* + v_2 x v_2^*$  with  $v_1 v_2^* + v_1^* v_2 = 1$ , going via a cocycle conjugation, we can assume without loss of generality that  $v_1 = D_1$  is a diagonal matrix with non-negative entries and  $v_2 = U D_2$  where  $D_1, D_2$  are diagonal matrices with non-negative entries and  $U$  is an element in  $SU(2)$  ( absorbing a phase factor i.e. replacing  $v_k$  by  $e^{i\theta} v_k$  ). The data matrix  $tr(v_i v_j^*)$  remains invariant under conjugation by unitaries and so without loss of generality we assume that  $tr(v_i v_j^*) = \delta_j^i \lambda_i$ . In particular now we get  $tr(UD_2 D_1) = 0$  i.e.  $\alpha D_1(1, 1) D_2(1, 1) + \bar{\alpha} D_1(2, 2) D_2(2, 2) = 0$ .

Case 1.  $\alpha \neq 0$ : Taking real and complex part, we get  $D_1(1, 1) D_2(1, 1) + D_1(2, 2) D_2(2, 2) = 0$  if  $Re(\alpha) \neq 0$  and  $D_1(1, 1) D_2(1, 1) - D_1(2, 2) D_2(2, 2) = 0$  if  $Im(\alpha) \neq 0$ . Unital property also ensures that for  $k = 1, 2$ ,  $D_k(1, 1)^2 + D_k(2, 2)^2 = 1$ . Now consider the function  $f(x) = x(1-x)$  on  $[0, 1]$  and note that in the later situation  $f(D_1(1, 1)^2) = f(D_2(1, 1)^2)$  and thus  $D_1(1, 1) = D_2(1, 1)$  and thus  $D_1 = D_2$ . In such that by Choi's criteria  $\tau$  is not extremal. Thus we are forced to situation where all entries are non-negative and  $D_1(1, 1) D_2(1, 1) = D_1(1, 1) D_2(2, 2) = 0$ . Thus  $U \in SU(2)$  with  $Re(\alpha) \neq 0$ ,  $Im(\alpha) = 0$  and either  $D_1 = |e_1 \rangle \langle e_1|$ ,  $D_2 = |e_2 \rangle \langle e_2|$  or  $D_2 = |e_2 \rangle \langle e_2|$ ,  $D_1 = |e_1 \rangle \langle e_1|$ . In such a case  $v_1 = |e_1 \rangle \langle e_1|$  and  $v_2 = \beta |e_1 \rangle \langle e_2| + \bar{\alpha} |e_2 \rangle \langle e_2|$  where  $|\alpha|^2 + |\beta|^2 = 1$ . But it also shows that  $v_2 v_1^* = 0$ . Thus  $\tau$  is not an extremal element when  $\alpha \neq 0$ .

Case 2.  $\alpha = 0$ . Situation is quite simple. It says that  $v_1^* = c_1 |e_1 \rangle \langle e_1| + c_2 |e_2 \rangle \langle e_2|$  is a pure diagonal and  $v_2^* = d_1 |e_2 \rangle \langle e_1| - d_2 |e_1 \rangle \langle e_2|$  is pure off diagonal where  $c_1, c_2 \geq 0$  and  $d_1, d_2 \geq 0$  and where we have used the unitary transformation  $e_1 \rightarrow \beta e_1$  and  $e_2 \rightarrow e_2$  in order to absorb  $\beta \in C$  with  $|\beta| = 1$ , which does not change the orbit generated by the  $CP$  map  $\tau$ .

We also compute

$$v_2 v_1^* = U D_2 D_1 = -c_1 d_2 |e_2 \rangle \langle e_1| + c_2 d_1 |e_1 \rangle \langle e_2|$$

Thus  $\tau$  is an extremal element if these two sets:  $v_1 v_1^*, v_2 v_2^*$  are linearly independent

and  $v_1v_2^*, v_2v_1^*$  are linearly independent. Both gives same relation

$$d_1c_2 \neq d_2c_1$$

as condition for linear independence.

So orbit space is determined by  $\tau(x) = v_1xv_1^* + v_2xv_2^*$  where  $v_1 = c_1|e_1\rangle\langle e_1| + c_2|e_2\rangle\langle e_2|$  and  $v_2 = d_1|e_1\rangle\langle e_2| - d_2|e_2\rangle\langle e_1|$  and parameter space satisfies the unital relation

$$c_1^2 + d_1^2 = 1, c_2^2 + d_2^2 = 1 \quad (5.1)$$

with  $d_1c_2 \neq c_1d_2$   $c_1, c_2, d_1, d_2 \geq 0$

Without loss of generality we assume that

$$c_1^2 + c_2^2 \leq 1 \leq d_1^2 + d_2^2 \quad (5.2)$$

Further using symmetry without loss of generality we assume that  $c_1 < c_2$  and so  $d_2 < d_1$ . In such a case we ensure

$$0 < c_1d_2 - c_2d_1 \neq 0 \quad (5.3)$$

Thus  $\tau$  is completely determined by  $\alpha = c_1$  and  $\beta = c_2$  with range  $0 \leq \alpha < \beta \leq 1$  and each distinguish element will have non-conjugate orbits under  $\mathcal{G}_0$ .

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